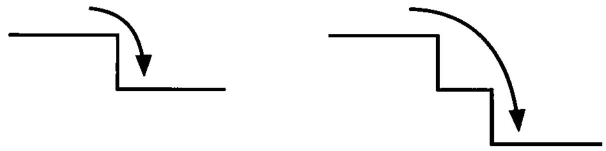
Chapter 1.

True or false?

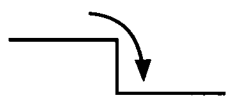
Situation: Stairs.

Let us suppose that the descent of a set of stairs can be made by taking the stairs

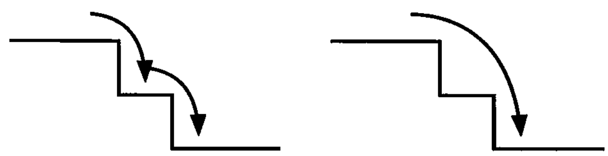
one at a time or two at a time.



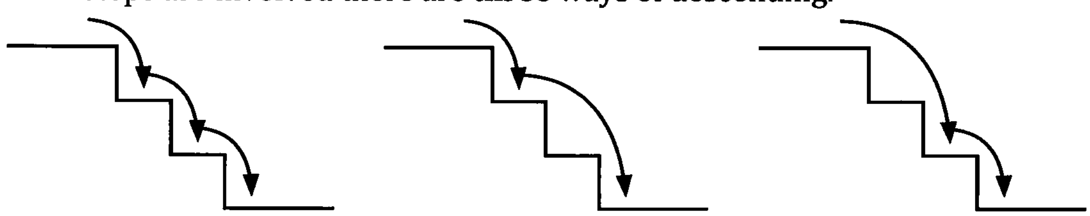
If the stairs involve just **one** step there is just **one** way of descending – one step at a time.



If the stairs involve **two** steps there are **two** ways of descending.



If **three** steps are involved there are **three** ways of descending.



Seeing the above pattern John conjectures* that if we use "one step at a time" or "two steps at a time" then there will be

Four ways of descending stairs with four steps.
Five ways of descending stairs with five steps.
Six ways of descending stairs with six steps.
Etc.

Is John's conjecture correct? If you think it is not correct explain why you think this and try to come up with a correct conjecture yourself for this situation.

* A *conjecture* is an opinion. It is what someone is suggesting they think to be the case. It is likely that John has formed this opinion based on observation and thought - in this case by observing and thinking about the pattern of numbers from the 1-step, 2-step and 3-step situations.

True or false?

Consider each of the following statements and for each one decide if the statement is true or false.

Note: In this activity only accept something as being true if there are no circumstances in which it can be false. Thus if $x^2 = 16$ the conclusion that $x = 4$ will be adjudged to be false because there is a circumstance when x does not have to be 4, it could be −4. Hence the statement: *If* $x^2 = 16$ *then* $x = 4$ should be adjudged to be a false conclusion because x is not necessarily 4.

All prime numbers are odd numbers.

All right triangles are isosceles.

For all people, getting older is inevitable.

If a quadrilateral has equal length diagonals then the quadrilateral must be a rectangle.

There exists an even prime.

There exists an odd prime.

For all real numbers, the square of the number is non-negative, i.e. for all $x \in \mathbb{R}$, $x^2 \geq 0$.

For all people in Australia the equator is to the North.

If a particular animal is an eagle then the animal is a bird.

If $x^2 = 4$ then $x = 2$.

If a particular animal is a bird then the animal is an eagle.

If $x = 2$ *then* $x^2 = 4$.

Every living dog has a heart.
My cat has a heart
Therefore my cat is a dog.

A triangle with angles of 20°, 80° and 80° must be an isosceles triangle.

An isosceles triangle must have angles of 20°, 80° and 80°.

If a quadrilateral is cyclic then a circle can be drawn through its four vertices.

If $x^2 \neq 4$ then $x \neq 2$.

If a circle can be drawn through the four vertices of a quadrilateral then the quadrilateral is cyclic.

If a triangle has two sides of the same length then it has two angles equal in size.

If a triangle has two angles of the same size then it has two sides of the same length.

If a polygon has exactly four sides then the polygon is a quadrilateral.

If a polygon is not a quadrilateral then it does not have exactly four sides.

If a postman has seven letters to deliver to six letter boxes at least one letter box must get more than one letter.

If it is not "not raining" then it must be raining.

You cannot have a right triangle with one side of length $3x$ cm another side of length $(4x + 5)$ cm and the longest side of length $(5x + 4)$ cm.

Let us now recall the situation at the beginning of this chapter which involved the various ways we could negotiate a set of stairs consisting of various numbers of steps which we could take "one step at a time" or "two steps at a time".

You might recall that, having spotted a pattern, John made the conjecture that there were:

Four ways of descending stairs with four steps.
Five ways of descending stairs with five steps.
Six ways of descending stairs with six steps. Etc.

Did you agree with the conjecture?

If you checked the four step situation (or indeed any of the other situations after that of three steps) you should have found that the facts contradicted John's conjecture. Because John's conjecture, as stated above, makes a claim that applies to all situations of four steps or more we need find only one **counter example** to show the general conjecture to be false.

To test the validity of a generalisation we could systematically check specific cases to see if the statement holds true for them. The more examples we find for which the statement holds the more confident we might become with regard to the validity of the statement. However we need only find one **counter example** to show the generalisation to be false.

Did you use counter examples to convince yourself that the following statements on the previous two pages were not true?

All prime numbers are odd numbers.
All right triangles are isosceles.
If a quadrilateral has equal length diagonals then the quadrilateral must be a rectangle.

If we are making a statement about **all** members of a set then the statement must apply to **all** members of that set. Words like "all" or "many" are *quantifiers* - they indicate quantity. If we claim that something is true for **all** members of the set then just one counter example proves our claim wrong. The following statements on the previous two pages involved the word all but in each case they were correct as the statement did indeed apply to all members of the set:

For all people, getting older is inevitable.
For all real numbers, the square of the number is non-negative, i.e. for all $x \in \mathbb{R}$, $x^2 \geq 0$.
For all people in Australia the equator is to the North.

(The symbol $\forall$ can be used to represent "for all". Thus: $\forall$ real numbers x, $x^2 \geq 0$.)

Another quantifier would be the phrase "at least one". The following statements from the previous two pages used the idea that there was at least one of something by using the phrase "**there exists**".

There exists an even prime.
There exists an odd prime.

For such statements we only have to show the existence of one such item and the statement is true.

(The symbol $\exists$ can be used to represent "there exists". Thus: $\exists$ an even prime.)

Some statements are true **by definition.** If we define a cyclic quadrilateral to be one for which the four vertices lie on a circle it follows that a circle can be drawn through the four vertices of a cyclic quadrilateral.

Some of the statements that you were asked to consider as to whether they were true or false suggested that "if" something was true "then" this **implied** that something else was true.

For example: *If* $x = 2$ *then* $x^2 = 4$.

If a particular animal is an eagle then the animal is a bird.

Both of which are true.

If event P implies event Q, then we write "**If P then Q**" or, using the symbol "⇒" we can write P ⇒ Q. Thus $x = 2 \Rightarrow x^2 = 4$

and an animal is an eagle ⇒ the animal is a bird

The **converse** of "If P then Q" is "If Q then P", i.e. Q ⇒ P. However, just because P ⇒ Q it does not automatically follow that Q ⇒ P.

For example: *If* $x = 2$ *then* $x^2 = 4$ is true

but the **converse**: *If* $x^2 = 4$ *then* $x = 2$ is false (because x could be -2).

Similarly, from the statements given earlier, it may be true that every living dog has a heart but I cannot conclude that just because my cat has a heart it too must be a dog. Neither can I conclude from the fact that an eagle is a bird that a bird is necessarily an eagle. Similarly a triangle with angles of 20°, 80° and 80° must indeed be an isosceles triangle but one counter example would be sufficient to prove that the converse, i.e. that an isosceles triangle must have angles of 20°, 80° and 80°, is not the case.

The converse of a true statement need not be true.

However if it is the case that P ⇒ Q and Q ⇒ P the symbol ⇔ can be used, i.e. P ⇔ Q. Statements P and Q are then said to be **equivalent**.

For example:

If a triangle has two sides of the same length then it has two angles equal in size.

I.e.: two sides of triangle the same length ⇒ two angles of triangle equal.

If a triangle has two angles of the same size then it has two sides of the same length.

I.e.: two angles of triangle equal ⇒ two sides of triangle the same length.

Hence: Two sides of triangle the same length ⇔ two angles of triangle equal.

The statement, P ⇔ Q, can also be written as "P **if and only if** Q" (or as "P iff Q"):

A triangle has two sides of the same length if and only if it has two angles equal in size.

The **contrapositive** of "If P then Q" is "If not Q then not P".

Consider the true statement *If* $x = 2$ *then* $x^2 = 4$.

The contrapositive statement is *If* $x^2 \neq 4$ *then* $x \neq 2$. Also a true statement.

The contrapositive of a true statement is also true.

For example:

If a polygon has exactly four sides then the polygon is a quadrilateral.

is a true statement, as is the contrapositive:

If a polygon is not a quadrilateral then it does not have exactly four sides.

Some of the statements that you were asked to judge as being true or false were probably quite obviously true such as the statement about the postman with seven letters to post in six letter boxes. However the logic behind it, in this case called the **pigeon-hole principle**, can be useful.
The pigeon-hole principle states:

If there are n pigeon holes, n ≥ 1, and n + 1 pigeons to go in them, then at least one pigeon hole must get two or more pigeons.

Some of the statements involved the **negation** of a statement. If P is the statement

It is raining.

Then the negation of P is the statement

It is not raining.

The statement "*it is not not raining*" given on an earlier page involves a double negative. If it is not not raining it must be raining.

Some statements are not so immediately obvious as being true or false and require careful thought. How for example did you decide upon the truth or otherwise of the statement:

You cannot have a right triangle with one side of length 3x cm another side of length (4x + 5) cm and the longest side of length (5x + 4) cm.

One approach is to assume the opposite, i.e. assume that we can indeed have a right triangle with the given side lengths and then prove that this assumption leads to something that cannot be true. This is called **proof by contradiction** – we assume what we are asked to prove is not the case, and then show that this assumption leads to a contradiction.

Converse, inverse and contrapositive.

The statement "if P then Q" also has an **inverse** statement which is "if not P then not Q".
For example, the inverse of the statement: If $x = 2$ then $x^2 = 4$
Is If $x \neq 2$ then $x^2 \neq 4$.
This inverse being false as the counter example $x = -2$ shows.

Thus for the statement: if P then Q,
the converse statement is if Q then P,
the inverse statement is if not P then not Q,
and the contrapositive statement is if not Q then not P.

The contrapositive statement involves both the effect of the converse, in its switch of P and Q, and of the inverse, with its negation of both P and Q.

If the original statement, if P then Q, is true then the contrapositive is also true but the converse and the inverse may not be.

(Readers may notice that the inverse is the contrapositive of the converse.)

Exercise 1A

Using the idea behind the pigeon-hole principle, what can be concluded for each of the situations described in numbers 1 to 7.

1. A teacher of *Mathematics Specialist* asked eight students each to choose one question to do from numbers 1 to 7 of this exercise.

2. The Singh triplets are all in year three at the same primary school and this school only has two year three classes.

3. Let us suppose that a particular genetic marker possessed by every human has two billion different variations and there are over six billion people in the world.

4. Peter has 12 socks in a drawer, six red socks and six blue socks but other than their colour the socks are indistinguishable. Peter reaches in and pulls out three socks.

5. On average the number of hairs on a young adult human's head is about 100000 and we would not expect to find any human with as many as one million hairs on their head. However there are over twenty million people in Australia.

6. If we count our ancestors as our parents, our parents parents, our parents parents parents etc then if we go back one generation we have 2 ancestors in that generation, go back two generations and we have 4 ancestors in that generation, go back three generations and we have 8 ancestors in that generation. If we go back 40 generations the number of ancestors we have is theoretically greater than the number of people that have ever lived.

7. Fifteen people at a gathering start to mix and mingle. Each person shakes hands with however many of the other 14 people that they wish to. (Shaking hands with oneself does not count!)
   (a) What is the largest number of different people a person in this group could shake hands with?
   (b) What is the smallest number of different people a person in this group could shake hands with?
   (c) Could one person have shaken hands with the largest number possible (i.e. the answer for part (a)) AND someone else in the group have shaken hands with the smallest number possible (i.e. the answer for part (b))?

The **converse** of "if P then Q" is "if Q then P".
Each of the following statements should be assumed to be true. Write the converse of each statement, state whether the converse is true or false, and write $P \Leftrightarrow Q$ or $P \not\Leftrightarrow Q$.

8. If a polygon has exactly three sides then the polygon is a triangle.

9. If Jenny is talking then her mouth is open.

10. If the animal is a platypus then it is a mammal.

11. If the car is out of fuel it will not start.

12. If points lie on the same straight line then they are collinear points.

The **contrapositive** of "if P then Q" is "if not Q then not P".
Each of the following statements should be assumed to be true. Write the contrapositive of each statement and then check that each contrapositive is also true.

13. If today is Thursday then tomorrow is Friday.

14. If a number is even then it is a multiple of two.

15. If a triangle is scalene then it has three different length sides.

16. If my sprinklers are on then my lawn is wet.

17. If it is not a school day then Armand does not get up before 8 am.

The **inverse** of "if P then Q" is "if not P then not Q".
For each of questions 18 to 22
(a) state whether the initial statement is true or false,
(b) write the inverse of the given statement,
(c) state whether the inverse statement is true or false.

18. If a polygon is a triangle then its angles add up to 180°.

19. If a positive integer has exactly two factors then it is a prime number.

20. If the car battery is flat then the car will not start.

21. If there are letters in my mail box then the postperson has been to our road.

22. If a number is even then it is a multiple of 4.

23. For the statement

If a polygon is five sided then the polygon is a pentagon

Write the converse statement, the inverse statement and the contrapositive statement and in each case state whether true or false.

24. For the statement

If a quadrilateral is a square then the four angles of the quadrilateral are all right angles.

Write the converse statement, the inverse statement and the contrapositive statement and in each case state whether true or false.

Prove each of the following using the method of proof by contradiction.

25. A triangle with side lengths of 8 cm, 9 cm and 10 cm is not right angled.

26. Prove that there are no integers p and q such that $6p + 10q = 151$.

27. Prove that if a and b are any two positive real numbers:

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

28. Prove that a triangle with one side of length $5x$, another of length $12x + 13$ and the longest side of length $13x + 12$, cannot be a right angled triangle.

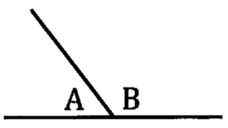
Miscellaneous Exercise One.

This miscellaneous exercise may include questions involving the work of this chapter and the ideas mentioned in the preliminary work section at the beginning of the book.

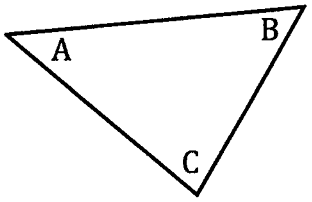
1. (Revision of right triangle trigonometry.)
A ladder stands with its base on horizontal ground and its top against a vertical wall. When the base of the ladder is a metres from the wall the ladder makes an angle of 75° with the ground. What angle will the ladder make with the ground if the base of the ladder is $\frac{5a}{4}$ metres from the wall?

Questions 2 to 8 (Revision of proofs using similar and congruent triangles).
Questions 2 to 8 each ask you to prove some geometrical fact. You should set out your proofs as in the examples given in the Preliminary Work sections involving similar triangles and congruent triangles. I.e. draw a clear diagram, state what you are given, what you have to prove and any constructions made. Then set out your proof clearly and with statements justified.
In your proofs the following may be stated as fact, without proof:

☛ Angles that together form a straight line have a sum of 180°.
I.e., in the diagram on the right, A + B = 180°.
(And, conversely, if angles have a sum of 180° then they together form a straight line.)



☛ The angles of a triangle add up to 180°.
I.e., in the diagram on the right, A + B + C = 180°.
(And conversely, if the angles of a polygon sum to 180° then the polygon is a triangle.)

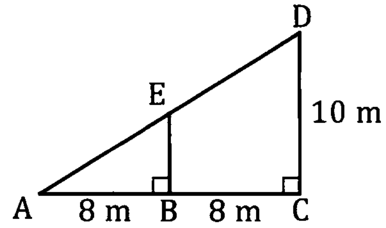


2. Prove that the angles of a quadrilateral add up to 360°.

3. In the diagram shown on the right,
E is a point on AD and B is a point on AC.

AB = 8 m
BC = 8 m
CD = 10 m

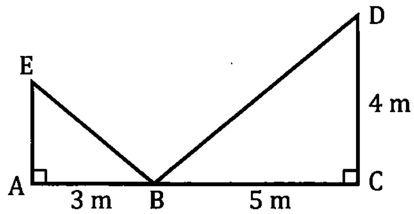
and right angles are as indicated.
Prove that EB = 5 m.



4. In the diagram shown on the right
B is a point on AC.

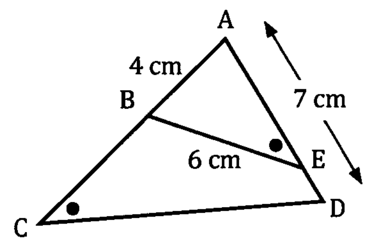
AB = 3 m
BC = 5 m
CD = 4 m.
∠EBD = 180° - 2 × ∠EBA

and right angles are as indicated.
Prove that AE = 2·4 m.



5. In the diagram shown on the right
B is a point on AC and E is a point on AD.

∠AEB = ∠ACD
AB = 4 cm
AD = 7 cm
and BE = 6 cm.
Prove that CD = 10·5 cm.



Each of questions 6, 7 and 8, involve an isosceles triangle. For each question assume only that the triangle has two sides of equal length. Any other facts already proved for isosceles triangles should not be assumed.

6. If we define an isosceles triangle to be a triangle that has two of its sides of the same length, prove that the line drawn from the vertex that is common to the equal sides, to the mid-point of the third side, is perpendicular to that third side.

7. ΔXYZ is an isosceles triangle with XZ = YZ.
Prove that if a line is drawn from Z to meet XY at right angles then this meeting point will be the mid-point of XY.

8. ΔPQR is an isosceles triangle with PQ = PR.
Prove that if a line is drawn from P, bisecting ∠QPR and meeting QR at the point M then M is the mid-point of QR.